Market Equilibrium Versus Optimum in a Model with Congestion: Note

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In their paper "Market Choice and Optimum City Size," Edwin Mills and David de Ferranti constructed a model of a city with congestion costs and derived the optimalland use pattern for it. However, they did not investigate the market equilibrium solution of their model and compare it to their optimal solution, as was done by Yitzhak Oron, David Pines, and Eytan Sheshinski.

In this paper the market equilibrium solution of the model is obtained and compared with the optimum solution derived by Mills and de Ferranti. It is shown that when congestion tolls are fully paid, market equilibrium is equivalent to optimality, as is the case in the paper by Oron et al. When no congestion tolls are paid, the city spreads further out than in the optimal case. The radius of the city increases, and land allocated to transportation at each location increases as well. Oron et al. were not able to obtain this result in their model.

Apart from the land use pattern, the average total cost of the urban economic activity in the model (housing and transportation) is investigated as a function of the population size. It is proved that average total cost and marginal total cost increase with population in the optimal case for any given population size. This result disproves the statement by Mills and de Ferranti that an optimum city size exists.

I. The Equilibrium Model

Mills and de Ferranti describe an optimization model of a town with congestion. Using their assumptions and notations, let u designate the distance from the center of the town. Then $L_1(u)$ is the land allocated to residential use at distance u, $L_2(u)$ the land allocated for transportation at distance u, and $T(u)$ the number of commuters crossing a circle of radius u. Let $p(u)$ be the cost per mile per commuter at distance u. Due to congestion this cost increases with the number of commuters $T(u)$ at distance u, and decreases with the land allocated for transportation, $L_2(u)$. We assume, following Mills and de Ferranti and William Vickrey, that it has the functional form as in equation (1).

$$(1) \qquad p(u) = \bar{p} + \rho_1 \left[\frac{T(u)}{L_2(u)} \right]^{\rho_2}$$

Let a_1 be the number of residents living on a unit of residential land. We assume a_1 to be constant. If $N(u)$ is the number of residents at distance u from the CBD, then

$$(2) \qquad N(u) = a_1 L_1(u)$$

Let θ, $0 \leq \theta \leq 2\pi$ be the section of town, measured in radians, available for residential and transportation activities, then (3) follows immediately.

$$(3) \qquad L_1(u) + L_2(u) = \theta u$$

$T(u)$, the number of commuters crossing a circle of radius u, is equal to the number of residents living outside this circle, hence

$$(4) \qquad T(u) = \int_u^{\bar{u}} N(u) du$$

where $\bar{u}$ is the radius of the town.

The problem Mills and de Ferranti solved is the minimization of the total social cost TC associated with the residential and commuting activity, where TC is given in (5):

$$(5) \qquad TC = \int_\epsilon^{\bar{u}} (p(u)T(u) + R_A \theta u) du$$

where ϵ, the radius of the CBD, and R_A, the agricultural land rent, are constants and

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exogenous given numbers.

The necessary condition for optimum is given by equation (6), (see (8) in Mills and de Ferranti):

$$(6) \qquad \frac{d\left[\frac{T(u)}{L_2(u)}\right]}{du} = -\frac{a_1}{\rho_2} - \frac{a_1\bar{p}}{\rho_1\rho_2(\rho_2+1)} \cdot \left[\frac{T(u)}{L_2(u)}\right]^{-\rho_2}$$

We will now proceed to investigate the market equilibrium case. Let $R(u)$ be the equilibrium rent function and $C(u)$ the price an individual pays when he crosses a circle with radius u. The total housing and transportation cost of an individual located at distance u from the center is then:

$$(7) \qquad \int_{\epsilon}^{u} C(x)dx + a_1^{-1}R(u)$$

where the first term expresses transportation cost and the second is the cost of the residential land. In this model no other costs are involved. This total cost of an individual should be the same everywhere under competitive equilibrium; otherwise, there will be excessive demand in some locations and no demand in others. Hence, by differentiating (7) with respect to u, we obtain

$$(8) \qquad C(u) + a_1^{-1}(dR(u)/du) = 0$$

A second condition is that land rents in each location be equal to the value of the marginal product of land in the transportation sector. The transportation product is the saving of costs rather than the production of positive income. The total value of transportation produced at location u is given by $(-T(u)p(u))$. The equality between the value of marginal product of land and rent is expressed in the following equation:

$$(9) \qquad R(u) = \partial(-T(u)p(u))/\partial L_2(u)$$

This equation implies that transportation is produced efficiently even if its product is not necessarily sold in the market.

Substituting from (1) for $P(u)$ in the above equation and then differentiating, we get

$$(10) \qquad R(u) = \rho_1\rho_2(T(u)/L_2(u))^{\rho_2+1}$$

We first assume that individuals pay full congestion tolls—i.e., each individual who passes through a circle with radius u pays an equal share of the rent of the land used for transportation at this radius. The total land used for transportation in radius u is $L_2(u)$, and the number of people crossing this circle is $T(u)$; hence, the equal share that each individual pays is $R(u)L_2(u)/T(u)$. Then $C(u)$ with full congestion tolls paid is:

$$(11) \qquad C_o(u) = p(u) + R(u)L_2^o(u)/T_o(u)$$

where the index o indicates the case with congestion tolls. Substituting (11) into (8), and then substituting in it $R(u)$ from (10) and $dR(u)/du$ by differentiating (10), we get

$$(12) \qquad \frac{d\left[\frac{T_o(u)}{L_2^o(u)}\right]}{du} = -\frac{a_1}{\rho_2} - \frac{a_1\bar{p}}{\rho_1\rho_2(\rho_2+1)} \cdot \left[\frac{T_o(u)}{L_2^o(u)}\right]^{-\rho_2}$$

But this is exactly equation (6) which is also equation (8) in Mills and de Ferranti. This means that the solution to the optimal problem solved by Mills and de Ferranti is also a solution to the case of market equilibrium with full congestion tolls.

Let us now assume that no congestion tolls are paid, i.e.,

$$(13) \qquad C_e(u) = p(u)$$

where e indicates the case without congestion tolls. We still assume efficiency in transportation production; i.e., (9) and (10) are still valid. If we substitute (13) into (8) and substitute for $dR(u)/du$ from (10) and for $p(u)$ from (1), we get

$$(14) \qquad \frac{dy_e}{du} = -\frac{a_1}{\rho_2(\rho_2+1)} - \frac{a_1\bar{p}}{\rho_1\rho_2(\rho_2+1)} y_e^{-\rho_2}$$

where:

$$y = T(u)/L_2(u) \tag{15}$$

and y_e is y in the case with no congestion tolls.

We will now try to compare characteristics of the town with congestion tolls to those of the town without congestion tolls. By setting $R(\bar{u}) = R_A$ in (10), we get:

$$y_o(\bar{u}_o) = y_e(\bar{u}_e) \tag{16}$$

From (12) and (14) we learn that the decline in $y_o(u)$ when u decreases is greater than in $y_e(u)$; i.e., y_o increases more rapidly with $(\bar{u}-x)$ than y_e. Hence, for any given distance from the boundary of the city x,

$$y_o(\bar{u} - x) > y_e(\bar{u} - x)$$

This means that at the distance x from the boundary of town,

$$T_o(\bar{u}_o - x)/\overset{o}{L_2}(\bar{u}_o - x) > T_e(\bar{u}_e - x)/\overset{e}{L_2}(\bar{u}_e - x)$$

This implies that at least one of the following relations must always be fulfilled.

$$T_e(\bar{u}_e - x) < T_o(\bar{u}_o - x) \tag{17a}$$

$$\overset{e}{L_2}(\bar{u}_e - x) > \overset{o}{L_2}(\bar{u}_o - x) \tag{17b}$$

If (17b) is fulfilled, it means that at any location with equal distance from the radius of town, more land is allocated for transportation in the town without congestion tolls (town e) than in the one with congestion tolls (town o). Less land will be allocated for dwelling at each such location in town e and therefore fewer residents will live there (than in town o). Equation (17a) is thus implied. If, on the other hand, we assume (17a), then between the town boundary and any given distance x from the boundary, fewer residents will live in town e than in town o. This implies that less land for dwelling and more land for transportation is allocated at this location in town e than in town o, which implies (17b). Hence, both relations in (17) must be fulfilled simultaneously. This implies that if both towns are of the same population size, town e must be more spread out than town o, i.e.,

$$\bar{u}_e > \bar{u}_o \tag{18}$$

The reason for this is that (17) implies that at any given distance x from $\bar{u}$, the density of population in town e is lower than in town o. For the two towns to be of the same size, ϵ must be further away from $\bar{u}_e$ than from $\bar{u}_o$ which implies (18).

The economic reasoning behind this argument is straightforward. Since land for transportation is a free product in the case without congestion tolls, it is abundantly used—so at every location there is too much land for transportation. Hence, the city is bigger than the optimal city size for the same population.

II. Average Total Cost

It is interesting to investigate the behavior of the total housing and transportation cost in this model as a function of population size. Mills and de Ferranti refer briefly to this question in the last section of their paper. They state that the average total cost for a low level of population size in the optimal town decreases when the population increases up to a certain minimum level of average total cost (ATC). A further increase in population size will cause ATC to increase. Hence, the city size in that minimum level of ATC is the optimum city size. This result has been questioned in the past,[1] since there is nothing in the model that permits economies of scale. It is proved in the following that ATC and marginal costs increase with population size at any level of population. It follows that no optimum city size exists.

Let us consider an optimal town with any given population N. We now reduce this population by an arbitrary quantity ΔN, $0<\Delta N<N$. The reduction will be done in a specific way and in two stages. The town at the final stage will be a feasible town, with total cost reduced more than proportionally to the reduction in population. It follows immediately that in an optimal town of the same population, the total cost will be even lower. Hence, marginal and average costs

[1] David Pines was the first to mention to me his doubts about this result. Mills himself also expressed, in a private communication, his doubts concerning this result.

increase monotonously at any given level of population.

Let us clarify the original optimal town with population of N by the subscript a. In the first stage, we reduce the population of town a by ΔN to get town b with a population $N-\Delta N$. The reduction is done by removing ΔN families living farthest from the center, and leaving the rest of the population intact. We then get a town with a new radius $\bar{u}_b$ and population $N-\Delta N$. Let us further reduce $L_2(u)$ at each $\epsilon<u<\bar{u}_b$ by the proportion $\Delta N/N$. The total cost of this new town, b, is less than a proportional reduction of the total cost of town a.

To prove it, let us consider the total cost function (5) and write it as follows:

$$(19)\qquad TC = \int_\epsilon^{\bar{u}} p(u)T(u)du + SR_A$$

where S is the total area belonging to the town between ϵ and $\bar{u}$. The term $p(u)$ is given by (1), with $\bar{p}=0$.

First we prove that $S_b<S_a(1-\Delta N/N)$ where S_i—total area of town i and $S_i=L_1^i+L_2^i$ and where:

$$(20a)\qquad L_1^i=\int_\epsilon^{\bar{u}_i} L_1^i(u)du$$

—total residential area in town i

$$(20b)\qquad L_2^i=\int_\epsilon^{\bar{u}_i} L_2^i(u)du$$

—total land for transportation in town i

$i=a, b, c$, is an index of town

$$L_1^b=\left(1-\frac{\Delta N}{N}\right)L_1^a$$

since residential land is proportional to population.

$$L_2^b(U) = (1-\Delta N \mid N)L_2^a(u) \qquad \text{for } \epsilon < U < \overline{U}_b$$

Hence,

$$L_2^b<\left(1-\frac{\Delta N}{N}\right)L_2^a$$

since $L_2^a(u)$ was reduced proportionally for $\epsilon\le u<\bar{u}_b$ and eliminated completely for $\bar{u}_b<u\le\bar{u}_a$. It follows that

$$(21)\qquad R_AS_b<\left(1-\frac{\Delta N}{N}\right)R_AS_a$$

From the definition of $T(u)$ in (4), it follows that

$$T_b(u) = T_a(u) - \Delta N$$

Putting $T_a(u)$ outside the brackets, we get

$$T_b(u) = T_a(u)\left(1-\frac{\Delta N}{T_a(u)}\right)$$

By substituting N for $T_a(u)$ in the denominator, we get, since $N\ge T_a(u)$

$$(22)\qquad T_b(u) \le T_a(u)\left(1-\frac{\Delta N}{N}\right)$$

and equality holds only when $T_a(u)$ equals N or zero.

From equation (1) with $\bar{p}=0$, we get

$$p_b(u) = \rho_1(T_b(u)/L_2^b(u))^{\rho_2}$$

Dividing the numerator and denominator by the factor $(1-\Delta N/N)$, and then substituting in the result $L_2^a(u)$ and $T_a(u)$, we get

$$\begin{aligned} p_b(u) &= \rho_1[(T_b(u)/(1-\Delta N/N) \\ &\qquad /(L_2^b(u)/(1-\Delta N/N))]^{\rho_2} \\ &\le \rho_1(T_a(u)/L_2^a(u))^{\rho_2} \end{aligned}$$

hence,

$$p_b(u) \le p_a(u)$$

From the above and (19), it follows:

$$(23)\qquad T_b(u)p_b(u) \le T_a(u)p_a(u)\left(1-\frac{\Delta N}{N}\right)$$

and equality holds only when $T_a(u)$ equals N or zero. Hence,

$$(24)\qquad \int_\epsilon^{\bar{u}_b} p_b(u)T_b(u)du < \left(1-\frac{\Delta N}{N}\right)\int_e^{\bar{u}_a} p_a(u)T_a(u)du$$

By summing up (21) and (24) and substituting (19) into the result, we get

$$(25) \quad TC(b) < (1 - \Delta N/N)TC(a)$$

However, our proof is not yet complete, since town b does not fulfill all the requirements of a town in the model; at every $\epsilon < u < \bar{u}_b$ in b, there is a greater part not belonging to the town than the $(2\pi - \theta)$ allowed. We therefore change town b into town c by moving people from the town limits into the surplus vacant land inside the town. This transfer once again reduces TC by reducing $T(u)$ and $\bar{u}$, i.e., $T_c(u) < T_b(u)$ and $\bar{u}_c < \bar{u}_b$, while L_1 and $L_2(u)$ remain unchanged. Hence,

$$(26) \quad TC(c) < TC(b) < (1 - \Delta N/N)TC(a)$$

Town c fulfills the requirements of a town in the model. This completes the argument.

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